7.4. Operational Properties II

Thm 1: If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(t^n f(t))$
$= (-1)^n \frac{d^n}{ds^n} F(s)$ ← (n-th derivative of F)

Proof: Part integration exchanges : t^n for n-th derivative

Ex: $f(t) = t \sin(2t)$
$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+4}$

Applying Thm 1 for $n = 1$, we have:

$$\mathcal{L}(f(t)) = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right) = (-1)\frac{2(-2s)}{(s^2+4)^2}$$

$$= \frac{4s}{(s^2+4)^2}.$$

Ex: $f(t) = t e^{2t} \cos t$
$\mathcal{L}(f(t)) = (-1)\frac{d}{ds}\left(\mathcal{L}(e^{2t}\cos t)\right)$ (account for t)

$$\mathcal{L}(e^{2t}\cos t) = \mathcal{L}(\cos t)\big|_{s-2} = \frac{s}{s^2+1}\Big|_{s-2} = \frac{s-2}{(s-2)^2+1}$$

$$\mathcal{L}(f(t)) = (-1)\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+1}\right) = \frac{1-(s-2)^2}{((s-2)^2+1)^2}.$$

Cor: $\mathcal{L}^{-1}(F(s)) = t^{-n}\mathcal{L}^{-1}\left((-1)^n \frac{d^n}{ds^n} F(s)\right)$
(If we know $\mathcal{L}^{-1}$ of some derivative of F then

we can find $\mathcal{L}^{-1}$ of F)

Ex: $F(s) = \ln\left(\frac{s+2}{s-5}\right)$, Find $\mathcal{L}^{-1}(F)$

Realize that $\frac{\partial F}{\partial s} = \frac{1}{s+2} - \frac{1}{s-5}$ (rational function)

$$\mathcal{L}^{-1}\left(\frac{\partial F}{\partial s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = e^{-2t} - e^{5t}$$

Applying Cor we have,

$$\mathcal{L}^{-1}(F) = t^{-1}\mathcal{L}^{-1}\left(-\frac{\partial F}{\partial s}\right) = t^{-1}\left(e^{5t} - e^{-2t}\right) \square$$

Thm 2: If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}$.

($\int_0^t f$ is like an anti-derivative of f)

Cor: $\int_0^t \mathcal{L}^{-1}(F(s))\,d\tau = \mathcal{L}^{-1}\left(\frac{F(s)}{s}\right)$.

Ex: $F(s) = \frac{1}{s(s-5)} = \frac{1/5}{s-5} - \frac{1/5}{s}$ (partial fraction)

Method 1: $\mathcal{L}^{-1}(F(s)) \overset{\text{Cor}}{=} \mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s-5}\right)$

$$= \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s-5}\right)d\tau = \int_0^t e^{5\tau}\,d\tau$$

$$= \frac{e^{5\tau}}{5}\Big|_0^t = \frac{e^{5t}-1}{5}.$$

<u>Method 2</u> : $\mathcal{L}^{-1}(F(s)) \overset{\text{Partial fraction}}{=} \mathcal{L}^{-1}\left(\frac{1/5}{s-5} - \frac{1/5}{s}\right)$

$$= \mathcal{L}^{-1}\left(\frac{1/5}{s-5}\right) - \mathcal{L}^{-1}\left(\frac{1/5}{s}\right) = \frac{1}{5}e^{5t} - \frac{1}{5}1.$$

$$= \frac{e^{5t}-1}{5}.$$

<u>Convolutions</u> :

<u>Def</u> : Given 2 functions $f(t)$ and $g(t)$ we define the convolution product $(f*g)(t) = \boxed{\int_0^t f(\tau)g(t-\tau)\,d\tau}$

<u>Rmk</u> : $\int_0^t f(\tau)\,d\tau = f * 1$

<u>Thm 3</u> : (convolution and Laplace transform)

1. $(f*g)(t) = (g*f)(t)$
2. $\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$!!!

($\mathcal{L}$ transforms a complicated product to a simple one)

<u>Ex</u> : $\int_0^t f(\tau)\,d\tau = f * 1$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) \overset{\text{Thm 3}}{=} \mathcal{L}(f)\cdot\mathcal{L}(1) = \frac{F(s)}{s}$$

<u>Cor</u> : $\mathcal{L}^{-1}(F(s)\cdot G(s)) = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G).$

<u>Ex</u> : $F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s}\cdot\frac{1}{s^2+1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$$

By the Cor, $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$

$$= 1 * \sin = \int_0^t \sin\tau \, d\tau = -\cos\tau \Big|_0^t$$

$$= 1 - \cos t$$

<u>Ex</u>: $F(s) = \dfrac{1}{s^2(s^2+1)}$,

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)$$

$$= 1 * (1 - \cos\tau) = \int_0^t (1-\cos\tau)\, d\tau$$

$$= (t - \sin\tau)\Big|_0^t = \boxed{t - \sin t}$$

<u>Ex</u>: Solving an IVP:

$$y'' + y = 2\cos t, \quad \underbrace{y(0) = 0, \; y'(0) = 0}$$

$$\mathcal{L}(y'' + y) = 2\,\mathcal{L}(\cos t)$$

$$\mathcal{L}(y) = Y, \quad \mathcal{L}(y'') = s^2 Y - s^1 y(0) - s^0 y'(0)$$

$$= s^2 Y \quad (y(0) = y'(0) = 0)$$

$$s^2 Y + Y = 2\frac{s}{s^2+1}$$

$$Y = \frac{2s}{(s^2+1)(s^2+1)}$$

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)(s^2+1)}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right)$$

$$= 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 2\cos * \sin = 2\int_0^t \boxed{\sin(t-\tau)\cos\tau}\,d\tau$$

Recall: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\Rightarrow \quad \sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$$= 2\,\frac{1}{2}\int_0^t \sin(t-\tau+\tau) + \sin(t-2\tau)$$

$$= \int_0^t \left[\sin t + \sin(t-2\tau)\right] d\tau$$

$$= (\sin t)\,\tau\Big|_0^t + \int_0^t \sin(t-2\tau)\,d\tau$$

$$= t\sin t + \boxed{\int_0^t \sin(t-2\tau)\,d\tau}$$

Change of variable: $u = t-2\tau$, $du = -2\,d\tau$

$$\int_0^t \sin(t-2\tau)\,d\tau = \int_t^{-t} \sin u \left(-\frac{1}{2}\right) du$$

$$= +\frac{1}{2} \cos(u) \Big|_t^{-t} = 0.$$

Final answer $y = \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t\sin t.$

Periodic functions: $\{$NOT cover

7.5 (Dirac-Delta)$\}$